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COMMENT

Generalisations and randomisation of the plane Koch curve

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Abstract. The Koch curve evolves from a base equilateral triangle by the trisection of each side and the replication of the original triangle on the mid-section, the process being repeated *ad infinitum* by the addition of sets of successively smaller trangles. This process is generalised to replace the trisectioning by $(2k+1)$-sectioning. It is shown that a square is the only other regular polygon on which the $(2k+1)$-sectioning procedure can be implemented. The Koch curves thus generated are strictly self-similar, their fractal dimensions being similarity dimensions and enclose simply connected areas. Randomisation of the generating procedure is also discussed.

The Koch curve (von Koch 1903/4, 1906) is one of the earliest examples of the so-called *monster* curves: while it is of infinite length, it encloses a simply connected region of finite area in the xy plane. The method of constructing the Koch curve is as follows (Mandelbrot 1983). Take an equilateral triangle and trisect each of its sides; then, on the middle segment of each side, construct equilateral triangles whose interiors lie external to the region exclosed by the base triangle and delete the middle segments of the base triangle. This basic construction is then repeated on all of the sides of the resulting curve, and so on *ad nauseaum*. The curve is defined so that the areas of all triangles lie inside it and it should contain nothing else; the perimeter is the length of this curve. If $r=0$ denotes the stage of evolution when only the base triangle of unit side is there, $r=1$ denotes the stage when three triangles of side $\frac{1}{3}$ have been added, etc, then it can be easily seen that the perimeter P_r of the Koch curve at the rth stage is

$$P_r = (4/3)^r P_0.$$

The perimeter grows unboundedly as r increases, but the area A_r included by the curve at the rth stage is

$$A_r = A_0\left(1+\tfrac{3}{4}\sum_{i\in\{1,2,\ldots,r\}} 4^i 3^{-2i}\right).$$

As $r\to\infty$, $A_r\to\frac{8}{5}A_0$ and is very much finite. Now, the number of the straight line segments making up the curve at the rth stage is four times that of the line segments making up the curve at the $(r-1)$th stage, but the segments become three times smaller in size. Hence, the fractal (similarity) dimension of the Koch curve turns out to be $D=\log(4)/\log(3)$. An alternative construction of the Koch curve is given by Singh (1927) where it has been shown that the Koch curve corresponds to a multiply valued function $y=K(x)$, which can be parametrically expressed (Kaufmann 1931) as $x=\xi(t)$, $y=\psi(t)$, both $\xi(t)$ and $\psi(t)$ being single-valued and continuous functions of the parameter t.

The point to be noted here is that an equilateral triangle is simply a regular *tri*gon. Thus if, in the construction procedure of the Koch curve, the 'equilateral triangle(s)' are replaced by 'regular n-gon(s)', a set of generalised fractal curves may be obtained. Furthermore, the 'trisection' of each side can be replaced by other equipartitioning schemes. Even more generalised Koch curves may then result. These are the possibiliities we will explore in this comment.

First of all, let us simply replace the equilateral triangles by regular n-gons, while the evolution of the curve still takes place by trisecting each straight-line segment. The angle included by the two consecutive sides of the base n-gon must be $\leqslant\pi/2$, or else a simply connected curve would not result as the construction procedure continues. This restricts the choice of the n-gon to either an equilateral triangle or a square. Next, each side of the base triangle or square may be partitioned into $(2k+1)$ segments of equal size, $k\geqslant 1$, consecutively numbered $1, 2, \ldots, 2k+1$, and the replication process should be implemented on the even-numbered segments. Parenthetically, we note that partitioning into an even number of segments would lead to the evolution of multiply connected curves.

We will now compute the fractal dimension $D_{n,k}$, the perimeter $P_{r,n,k}$ and the enclosed area $A_{r,n,k}$ of the generalised Koch curves for $r=1,2,\ldots$, $n=3,4$ and $k=1,2,\ldots$. Let us consider, first, the case in which the basic polygon is an equilateral triangle ($n=3$). For a given k, the number of the straight-line segments making up the curve at the rth stage is $(k+1+2k)$ times that of the line segments making up the curve at the $(r-1)$th stage, but which are $(2k+1)$ times smaller in size. Hence, the fractal (similarity) dimension of the Koch curve turns out to be $D_{3,k}=\log(3k+1)/\log(2k+1)$, which is a number greater than unity. At the same time, the perimeter $P_{r,3,k}$ is related to $P_{0,3,k}$ by the relation

$$P_{r,3,k}=[(3k+1)/(2k+1)]^r P_{0,3,k}$$

from where it is obvious that for a given $(2k+1)$-equisectioning, the perimeter grows unboundedly with r. The area at the rth stage, however, is

$$A_{r,3,k}=A_{0,3,k}\left(1+3k(3k+1)^{-1}\sum_{i\in\{1,2,\ldots,r\}}(3k+1)^i(2k+1)^{-2i}\right)$$

which, in the limit $r\to\infty$, goes to $A_{\infty,3,k}=A_{0,3,k}4(k+1)(4k+1)^{-1}$.

Finally, let us consider the case in which the basic polygon is a square ($n=4$). For a given k, the number of the straight-line segments making up the curve at the rth stage is $(k+1+3k)$ times that of the line segments making up the curve at the $(r-1)$th stage, but which are also $(2k+1)$ times smaller in size. Hence, the fractal (similarity) dimension of the Koch curve turns out to be $D_{4,k}=\log(4k+1)/\log(2k+1)$, which is a number greater than unity. At the same time, the perimeter $P_{r,4,k}$ is related to $P_{0,4,k}$ by the relation

$$P_{r,4,k}=[(4k+1)/(2k+1)]^r P_{0,4,k}$$

from where it is clear that the perimeter grows unboundedly with r. The area at the

rth stage, however, is

$$A_{r,4,k} = A_{0,4,k}\left(1+4k(4k+1)^{-1}\sum_{i\in\{1,2,\ldots,r\}}(4k+1)^i(2k+1)^{-2i}\right)$$

which, in the limit $r\to\infty$, goes to $A_{\infty,4,k} = A_{0,4,k}(k+1)k^{-1}$. Shown in figure 1 are two samples of the Koch curves generated on a base square; the effect of the specific value of k selected is all too apparent.

In figure 2 we have plotted the computed values of $D_{n,k}$ and $A_{\infty,n,k}/A_{0,n,k}$ for $n=3$, 4 and $k=1,2,\ldots,15$. From this figure it is clear that as $k\to\infty$, $D_{n,k}\to 1$, as should be expected. In that limit, the sides of the base n-gon would appear simply to gain some thickness as r increases. Furthermore, $A_{\infty,n,k}/A_{0,n,k}$ for both values of n also tend to unity as k increases, which also supports that conclusion.

The Koch curves described so far are rigorously self-similar in that they possess fixed similarity dimensions. Because of the $(2k+1)$-sectioning of the sides of a given curve, however, the generation process can be easily modified to yield self-affine Koch curves. The differences between self-similarity and self-affinity have recently become topics of research and we refer the interested reader to Mandelbrot (1985) and Lakhtakia *et al* (1986) for discussion. Here, we will confine ourselves to giving an algorithm for generating 'randomised' Koch curves. In order to do so, we begin with either a base

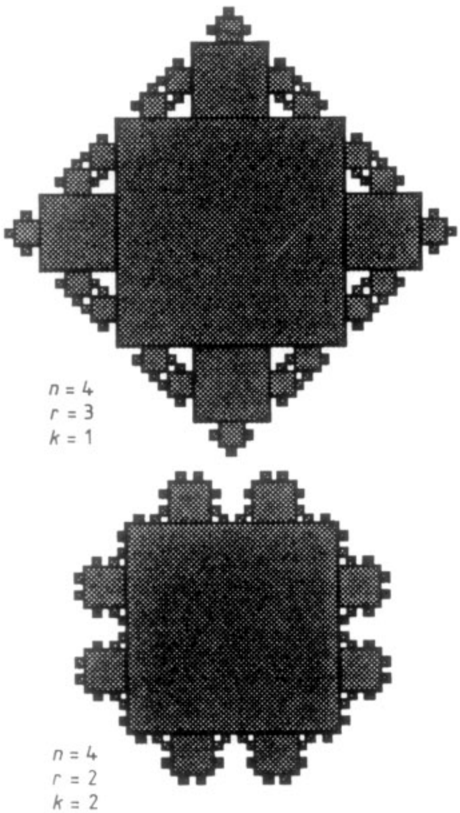

Figure 1. Two samples of the Koch curves generated on a base square with different values of k.

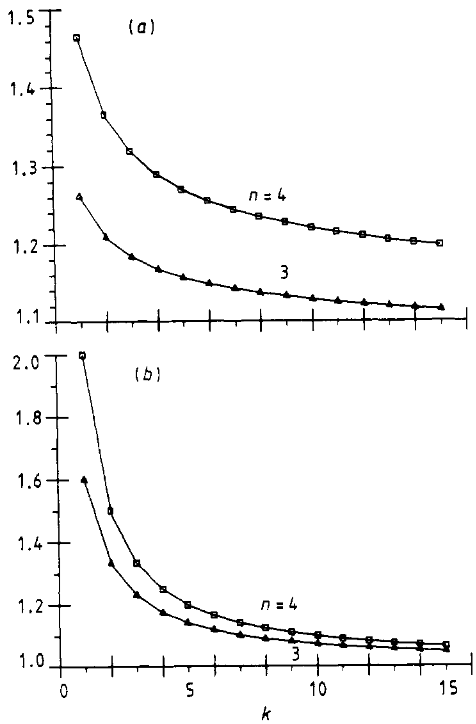

Figure 2. Computed values of (*a*) $D_{n,k}$ and (*b*) $A_{\infty,n,k}/A_{0,n,k}$ for $n=3,4$ against k for the generalised Koch curves.

equilateral triangle or a square and carry on the generating procedure described above; the exception being that the selected value of k for the $(2k+1)$-sectioning of the sides of the curve of stage r may not be a constant, depending, instead, on the stage r itself. In figure 3, a simple example of such a 'randomised' Koch curve is shown which starts from a base square; $k=1$ for $r=0,2$, and $k=2$ when $r=1$. Although due to the resolution limitations of the Macintosh Plus monitor and the MacDraw software, the generation process could not be carried on for long enough, it should be noted that the generated curve appears to have traits inherited from both of the examples shown in figure 1.

The visual similarity between figures 1 and 3 and crystal morphologies for cubic crystals, such as lead sulphide prepared under a range of gel preparation conditions (Garcia-Ruiz 1986), is striking. That the evolution of the morphologies (from dendrites to cubes) can result from the ordered aggregation of small cubic building blocks, as seen directly by scanning electron microscopy, suggests that the methods described above could have direct application. In particular, it has been shown (Gracia-Ruiz 1986) that, as the lead sulphide crystals grow, there is a change in the pH of the gel growth medium which leads to a continuous change in the stacking of the 'block nuclei'. This same growth mechanism has also been inferred for potassium chloride crystals grown from solution (Glasner and Tassa 1974a, b).

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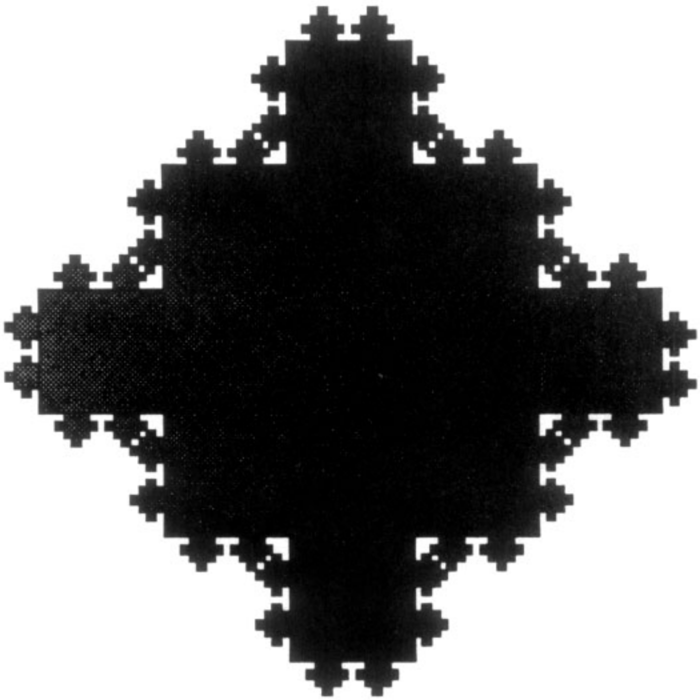

Figure 3. A 'randomised' Koch curve generated on square ($n = 4$) at stage $r = 3$. The sides of the curve are (i) three-sectioned when $r = 0$, (ii) five-sectioned when $r = 1$ and (iii) three-sectioned when $r = 2$.

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